Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 4, pp. 583–591, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)00085-9

A LINEAR EVASION PROBLEM FOR INTERACTING GROUPS OF OBJECTS†

P. V. PROKOPOVICH and A. A. CHIKRII

Kiev

(*Received* 15 *April* 1993)

At least one of m objects evades n pursuers in Euclidean k-space. All the objects are of the same type, and their motion is governed by linear dynamics. When considering this problem an auxiliary problem arises: evasion from fixed positions in a semi-infinite time interval.

The problem of evasion from given initial positions has previously been studied over a finite time interval [1]. The research reported here is closely related to that of [2–6].

1. Let R^k be Euclidean k-space, (x, y) the scalar product of vectors x and y of R^k, and $\|x\|=\sqrt{(x, x)}$. We let int X, ∂X, co X, con X denote the interior, boundary, convex hull and conical hull, respectively, of an arbitrary set $X\subset R^k$; $\Omega(R^k)(\text{co}\,\Omega(R^k))$ is the space of all non-empty compact sets (convex compact sets) in R^k with the following Hausdorff metric $S=\{x\in R^k: \|x\|\leqslant 1\}$; $N_q=\{1, 2, \ldots, q\}$. If X is a finite set, we let $|X|$ denote the number of its elements.

Given a set $F\in\Omega(R^k)$, we define the support function $c(F, \cdot): R^k\to R$ by

$$c(F,\psi)=\max_{f\in F}(f,\psi),\quad \psi\in R^k$$

Let $\psi_0\in R^k$, $\|\psi_0\|\neq 0$. The set

$$U(F,\psi_0)=\{f\in F:(f,\psi_0)=c(F,\psi_0)\}$$

is called the support set of F in the direction ψ_0. If the support set $U(F, \psi_0)$ is a singleton, we say that F is strictly convex towards $\psi_0\in R^k$. We shall say that a set $F\in\Omega(R^k)$ is strictly compact if it is strictly compact towards any $\psi_0\in R^k$, $\|\psi_0\|\neq 0$ [7]. A set $F\in\Omega(R^k)$ is said to be compact with smooth boundary if

$$U(F,\psi)\cap U(F,\psi')=\varnothing,\ \forall\psi,\psi'\in\partial S,\ \psi\neq\psi'$$

Note that a compact set with smooth boundary, like a strictly convex compact set, need not be a convex set. For example, the unit sphere ∂S in R^k is a strictly compact set with smooth boundary.

Consider a controllable object whose motion is described by a linear differential equation

$$\dot y=Ay+\upsilon,\quad \upsilon\in V,\quad V\in\Omega(R^k) \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 4, pp. 12–21, 1994.

where y is the k-dimensional phase state vector of the object, υ is a k-dimensional control vector, and A is a square matrix of order k. By an admissible control over an interval $I=[0,\ t_1]$ we mean any measurable function υ: $I\to V$.

Let $X(t_1;\ G, V)$ denote the set of admissibility for the controllable object (1.1) at time $t_1\geq 0$ from the initial set $G\in\Omega(R^k)$, i.e.

$$X(t_1;G,V)=\exp(t_1A)G+\int_0^{t_1}\exp((t_1-s)A)Vds$$

Let $y(t)$ be a solution of Eq. (1.1) corresponding to the control $\upsilon(t)$ and initial condition $y(0)\in G,\ G\in\Omega(R^k)$. We shall say [8] that the pair $(\upsilon(t),\ y(t))$ satisfies a maximum condition over the interval I and a transversality condition on the set G if a solution $\psi(t)$ exists of the auxiliary adjoint system of equations

$$\dot{\psi}=-A^*\psi \tag{1.2}$$

with initial condition $\psi(0)\in\partial S$ such that the following conditions hold

1. $(\upsilon(t),\ \psi(t))=c(V,\ \psi(t))$ for almost all $t\in I$,
2. $(y(0),\ \psi(0))=c(G,\ \psi(0))$.

Lemma 1. Let $G\in\operatorname{co}\Omega(R^k)$. Then $y(t_1)$ is an element of the set $\partial X(t_1;\ G,\ V)$ for $t_1>0$ if and only if the pair $(\upsilon(t),\ y(t))$ satisfies a maximum condition over I and a transversality condition on G.

Lemma 2 [4]. Let $y_j(t)$ be a solution of Eq. (1.1) corresponding to a control $\upsilon_j(t)$ and an initial condition $y_i(0)\in G,\ G\in\Omega(R^k)$, where the pair $(\upsilon_j(t),\ y_j(t))$ satisfies a maximum condition over the interval $I=[0,\ t_1],\ t_1>0$, and a transversality condition on $G\in\Omega(R^k)$; $\psi_j(t)$ is the corresponding solution of the adjoint system (1.2), $j=1,\ 2$. If $y_1(0)\neq y_2(0)$ and for at least one $j\in N_2$ and almost all $t\in I$, the function $c(V,\ \psi)$ is differentiable at the point $\psi_j(t)$. Then $y_1(t_1)\neq y_2(t_1)$.

Lemma 3 [5, 6]. If V is a compact set with smooth boundary, $G\in\operatorname{co}\Omega(R^k)$, then the set $X(t_1;\ G, V)$ where $t_1>0$, is a convex compact set with smooth boundary.

2. The motion of objects in R^k $(k\geq 2)$ is described by equations

$$P_i:\dot{x}_i=Ax_i+u_i,u_i\in U_i;\ \ E_j:\dot{y}_j=Ay_j+\upsilon_j,\ \ \upsilon_j\in V \tag{2.1}$$
$$U_i,V\in\Omega(R^k),\ U_i\subset\operatorname{co}V;\ i=1,\ldots,n,\ \ j=1,\ldots,m$$

with initial conditions $x_i(0)=x_i^0,\ y_j(0)=y_j^0$, where

$$x_i^0\neq y_j^0,\ \ i=1,\ldots,n,\ \ j=1,\ldots,m \tag{2.2}$$

Here $x_i,\ y_j$ are the phase coordinates of the ith pursuer and the jth evader and A is a square matrix of order k. The players' controls are measurable functions $u_i:[0,\ +\infty)\to U_i,\ \ \upsilon_j:[0,\ +\infty)\to V$.

We shall say that evasion is possible in game (2.1) starting in the initial state $z^0=(x_1^0,\ldots,x_n^0,\ y_1^0,\ldots,y_m^0)$ (the local evasion problem is solvable) if the players E_j $(j=1,\ldots,m)$ have controls such that, for any controls of players P_i $(i=1,\ \ldots,n)$, $s\in\{1,\ldots,m\}$ exists such that $x_i(t)\neq y_s(t)$ for all $i\in N_n,\ t\in[0,\ +\infty)$. Moreover, at time t the values of the evaders' controls are constructed on the basis of information on the state actually achieved

$$z(t)=(x_1(t),\ldots,\ x_n(t),y_1(t),\ldots,\ y_m(t))$$

and the values of the pursuers' controls on the basis of any conceivable information.

If evasion is possible in game (2.1) from any initial state z^0 that satisfies inequalities (2.2), we shall say that the global evasion problem is solvable in game (2.1).

3. Let G be some non-empty subset of R^k. Given an initial state $z^0=(x(0),\ y(0))=(x_1^0,\ldots,x_m^0,\ y_1^0,\ldots,\ y_m^0)$, we define several sets of indices

$$I(x(0),G)=\{i\in\{1,\ldots,n\}:x_i^0\in G\} \tag{3.1}$$

$$J(y(0),G)=\{j\in\{1,\ldots,m\}:y_j^0\in G\} \tag{3.2}$$

$$J_0(y(0),\partial G)=\{j\in\{1,\ldots,m\}:y_j^0\in \partial G\} \tag{3.3}$$

where, if there exist

$$j_l\in J_0(y(0),\partial G),\quad l=1,\ldots,s,s>1,\quad j_1<j_2<\ldots<j_s \tag{3.4}$$

such that $y_{j1}^0=y_{j2}^0=\ldots=y_{j_s}^0$, we shall assume that $j_l\notin J_0(y(0),\ \partial G)$, $l=2,\ldots,s$.

Let $G\in \text{co}\Omega(R^k)$. Define a mapping $P(G,\cdot):R^k\setminus \text{int}G\to\Omega(R^k)$ by the formula

$$P(G,y)=\partial S\cap[\text{con}(y-G)]^*,\quad y\in R^k\setminus\text{int}G \tag{3.5}$$

where $[\text{con}(y-G)]^*$ is the cone adjoint to $\text{con}(y-G)$. Let $\psi_j(t,\ r_j)$ denote a solution of system (1.2) corresponding to the initial condition $\psi_j(0)=r_j$, $r_j\in P(G,\ y_j^0)$, $j\in J_0(y(0),\ \partial G)$.

Theorem 1. If a set $G\in\text{co}\Omega(R^k)$ exists such that $|J_0(y(0),\ \partial G)|>|I(x(0),\ R^k\setminus G)|$, and for any $j\in J_0(y(0),\ \partial G)$ and some $r_j\in P_j(G,\ y_j^0)$ the support function $c(V,\psi)$ is differentiable at the point $\psi_j(t,\ r_j)$ for almost all $t\in[0,\ +\infty)$, then evasion is possible in game (2.1) from the initial state z^0.

Proof. For any $j\in J_0(y(0),\ \partial G)$, choose a vector $r_j\in P(G,\ y_j^0)$ such that, for almost all $t\in[0,\ +\infty)$, the support function $c(V,\psi)$ is differentiable at $\psi_j(t,\ r_j)$.

As a control for player $E_j(j\in J_0(y(0),\ \partial G))$ we take a measurable function $\upsilon_j(t)\in V$, $t\geqslant 0$ such that

$$(\upsilon_j(t),\ \psi_j(t,r_j))=c(V,\psi_j(t,r_j)) \tag{3.6}$$

The control $\upsilon_j(t)$, $j\in J_0(y(0),\ \partial G)$ is uniquely defined by (3.6), because $c(V,\ \psi)$ is differentiable at $\psi_j(t,\ r_j)$ for almost all $t\geqslant 0$. The uniqueness of the control $\upsilon_j(t)$ is understood here in the sense that two measurable functions defined in one time interval are equal if their values are the same at almost every point of the interval. The controls of the evaders $E_j(j\in N_m\setminus J_0(y(0),\ \partial G))$ are arbitrary.

If follows from Lemmas 1 and 2 that a pursuer $P_i(i\in I(x(0),\ G))$ cannot capture any evader $E_j(j\in J_0(y(0),\ \partial G))$, while a pursuer $P_i(i\in I(x(0),\ R^k\setminus G))$ may capture at most one evader $E_j(j\in J_0(y(0),\ \partial G))$ in a semi-infinite time interval. Since $|J_0(y(0),\ \partial G)|>|I(x(0),\ R^k\setminus G)|$, this proves the theorem.

Note that differentiability of the support function $c(V,\ \psi)$ at a point $\psi_0\in R^k$, $\|\psi_0\|\neq 0$ implies that V is strictly convex toward ψ_0, and conversely.

Corollary 1. Suppose that in game (2.1) V is a strictly convex compact set and there exists $j\in N_m$ such that $y_j^0\notin\text{int}\,\text{co}\{x_1^0,\ldots,\ x_n^0\}$. Then the evasion problem is solvable from the initial state z^0.

This corollary generalizes a result of [9, 10], hitherto known to be true for simple motion, to linear systems.

Corollary 2. Suppose that in game (2.1) V is a strictly convex compact set, $n=k+1$ and $m=2$. Then the global evasion problem is solvable.

Proof. We may assume without loss of generality that $y_1^0 \neq y_2^0$. Let H denote a hyperplane passing through the points x_i^0, $i=1, \ldots, k-2$, y_1^0, y_2^0. If there are several such hyperplanes, choose one. Clearly, one of the open half-spaces determined by H contains either none of the points x_{k-1}^0, x_k^0, x_{k+1}^0 or only one of them. Consequently, a convex compact set G exists such that y_1^0, $y_2^0 \in \partial G$ and $|I(x(0), R^k \setminus G)| \leqslant 1$.

It follows from this assertion, in particular, that the global evasion problem for a game of three pursuers and two evaders is solvable in the plane for simple motion [3].

Corollary 3. Suppose that in game (2.1) V is a strictly convex compact set, $n=2k-1$ and $m=k$. Then the global evasion problem is solvable.

Corollary 4. Suppose that in game (2.1) V is a strictly convex compact set, $n=2k$ and $m=k$, The initial state $z^0=(x_1^0, \ldots, x_{2k}^0, y_1^0, \ldots, y_k^0)$ is such that $y_s^0 \neq y_q^0$ for any s, $q \in N_k$, $s \neq q$ and the initial positions of some $k+1$ players lie in one hyperplane.
Then the evasion problem from the initial state z^0 is solvable.

Theorem 2. Suppose that in game (2.1) V is a strictly convex compact set with smooth boundary. If sets G_1, $G_2 \in \mathrm{co}\Omega(R^k)$ exist such that $x_i^0 \in G_1 \cup G_2$ for any $i \in N_n$ and

$$|I(x(0),\ G_1 \setminus G_2)| < |J(y(0), R^k \setminus (G_1 \cup G_2))| + |J_0(y(0), \partial G_2)|$$

then the evasion problem is solvable from the initial state z^0.

Proof. Since $X(\delta; G_2, V)$ is a convex compact set with smooth boundary for arbitrarily small $\delta>0$, we may assume that G_2 has a smooth boundary. In that case the set $P(G_2, y_j^0)$ is a singleton for any $j \in J_0(y(0), \partial G_2$, containing the single point r_j. Define a control $\upsilon_j(t)$, $t \geqslant 0$, for any $j \in J_0(y(0), \partial G_2)$ by condition (3.6), letting $\psi_j(t, r_j)$ be a solution of system (1.2) corresponding to the initial condition $\psi_j(0)=r_j$. The controls of players $E_j(j \in N_m \setminus [J(y(0), R^k \setminus (G_1 \cup G_2)) \cup J_0(y(0), \partial G_2)])$ are defined arbitrarily.
The proof will proceed by induction on the number of evaders whose initial positions lie in the set $R^k \setminus (G_1 \cup G_2)$. We put $l=|J(u(0), R^k \setminus (G_1 \cup G_2))|$.
Consider the case $l=1$. For any $i \in I(x(0), \partial G_2)$, define a trajectory $\bar{x}_i(t)$, $t \geqslant 0$, which begins at a point x_i^0 and corresponds to a control $\bar{u}_i(t)$ chosen from the equality

$$(\bar{u}_i(t),\ \psi_i(t, \bar{r}_i)) = c(V, \psi_i(t, \bar{r}_i))$$

where $\psi_i(t, r_i)$ is a solution of system (1.2) with $\psi_i(0)=\bar{r}_i$, $\bar{r}_i \in P(G_2, x_i^0)$. Since V is strictly convex and G_2 is compact with smooth boundary, it follows that the trajectories $y_j(t)$, $j \in J_0(y(0), \partial G_2)$, $\bar{x}_i(t)$, $i \in I(x(0), \partial G_2)$, $t \geqslant 0$ are uniquely defined.
The control $\upsilon_j(t)$, $t \in [0, t(r_j))$, $j \in J(y(0), R^k \setminus (G_1 \cup G_2))$ will be determined by condition (3.6), in which $\psi_j(t, r_j)$ is a solution of system (1.2) corresponding to the initial condition $\psi_j(0)=r_j$, $r_j \in P(G_1, y_1^0)$, the vector r_j being chosen so that

$$y_j(t(r_j), r_j) \neq \bar{x}_i(t(r_j)),\ \ \forall i \in I(x(0), \partial G_2) \tag{3.7}$$

$$y_j(t(r_j), r_j) \neq y_s(t(r_j)),\ \ \forall s \in J_0(y(0), \partial G_2) \tag{3.8}$$

where $y_j(t, r_j)$ is the corresponding trajectory of player $E_j(j \in J(y(0), R^k \setminus (G_1 \cup G_2)))$, $t(r_j)$ and $t(r_j)$ is the first time at which $y_j(t, r_j) \in X(t; G_2, V)$. If $y_j(t, r_j) \notin X(t; G_2, V)$ for any $t \geqslant 0$, then by Lemmas 1 and 2 player $E_j(j \in J(y(0), R^k \setminus (G_1 \cup G_2)))$, employing such a control, will evade capture. We may therefore assume that $t(r_j) < +\infty$.
We shall now show that the vector $r_j \in P(G_1, y_j^0)$, $j \in J(y(0), R^k \setminus (G_1 \cup G_2))$, for which inequalities (3.7) and (3.8) hold, is unique. Indeed, since V is a compact set with smooth

boundary, it follows that

$$y_j(t,r_j^1) \neq y_j(t,r_j^2) \text{ for } t>0 \tag{3.9}$$

if $r_j^1,\ r_j^2 \in P(G_1,\ y_j^0),\ r_j^1 \neq r_j^2$.

Note that if it is true that, for some $r_j^1 \in P(G_1,\ y_j^0),\ j \in J(y(0),\ R^k \setminus (G_1 \cup G_2))$

$$y_j\big(t(r_j^1), r_j^1\big) = \bar{x}_s\big(t(r_j^1)\big) \tag{3.10}$$

for some $s \in I(x(0),\ \partial G_2)$, then for any other vector $r_j^2 \in P(G_1,\ y_j^0),\ r_j^1 \neq r_j^2$ (we are assuming that $t(r_j^2) < +\infty$) we have

$$y_j\big(t(r_j^2),\ r_j^2\big) \neq \bar{x}_s\big(t(r_j^2)\big) \tag{3.11}$$

Indeed, suppose the contrary

$$y_j\big(t(r_j^2), r_j^2\big) = \bar{x}_s\big(t(r_j^2)\big) \tag{3.12}$$

It follows from (3.9), (3.10) and (3.12) that $r(r_j^1) \neq t(r_j^2)$. We may assume without loss of generality that $t(r_j^1) < t(r_j^2)$.

Since

$$\bar{x}_s\big(t(r_j^1)\big) \in \partial X\big(t(r_j^1);\ y_j^0, V\big)$$

$$\bar{x}_s\big(t(r_j^2)\big) \in \partial X\big(t(r_j^2);\ y_j^0, V\big)$$

and (3.10) is true, it follows from Lemmas 1–3 that $\bar{x}_s(t(r_j^2)) = y_j(t(r_j^2),\ r_j^1)$, contradicting (3.9) This proves (3.11).

Thus, an evader $E_j(j \in J(y(0),\ R^k \setminus (G_1 \cup G_2)))$, knowing the initial positions of players $P_i(i \in I(x(0),\ \partial G_2))$, $E_q(q \in J_0(y(0),\ \partial G_2))$, will choose as $\psi_j(0)$ a vector $r_j \in P(G_1,\ y_j^0)$ such that, on the trajectory $y_j(t,\ r_j)$, $t \geqslant 0$, inequalities (3.7) and (3.8) hold at time $t = t(r_j)$.

Of course, whatever the controls of players P_i $(i = 1, \ldots, n)$ in the interval $[0,\ t(r_j)]$, at time $t = t(r_j)$, the state $z(t(r_j))$ will satisfy the assumptions of Theorem 1. As the set G in the formulation of Theorem 1 one can take $X(t(r_j);\ G_2;\ V)$.

Suppose that the assumptions of the theorem hold and that when $l \leqslant r$ evasion is possible from the initial state z^0 in game (2.1). We shall show that the statement of the theorem is true for $l = r+1$. Fix some set $F \in \operatorname{co}\Omega(R^k)$ such that $0 \in \operatorname{int} F$. We may assume that

$$J(y(0),\ R^k \setminus (G_1 \cup G_2)) = N_{r+1}$$

$$y_s^0 \in \partial(G_1 + \varepsilon_s F),\ \varepsilon_s > 0,\ s = 1, \ldots, r+1$$

where $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_{r+1} > \varepsilon_{r+2} = 0$. Indeed, if $q \in \{2,\ \ldots,\ r+1\}$ exists for which $\varepsilon_{q-1} = \varepsilon_q,\ \varepsilon_q > \varepsilon_{+1}$, then one can construct controls for the evaders E_j $(j = 1, \ldots, r+1)$ in a half-closed interval $[0,\ \delta)$ (where $\partial > 0$ is an arbitrary small number) so that

$$y_s(\delta) \in \partial(X(\delta; G_1, V) + \varepsilon_s \exp(\delta A)F),\ \ \forall s \in N_{r+1} \setminus \{q\}$$

$$y_q(\delta) \in \partial(X(\delta; G_1, V) + \varepsilon_q' \exp(\delta A)F),\ \ \varepsilon_{q-1} > \varepsilon_q' > \varepsilon_{q+1}$$

The control $\upsilon_j(t)(j \in N_{r+1})$ is defined from condition (3.6) in which $Gy_j(t,\ r_j)$ is a solution of system (1.2) corresponding to the initial condition $\psi_j(0) = r_j,\ r_j \in P(G_1 + \varepsilon_{j+1}F,\ y_j^0)$, where r_j is a vector satisfying (3.7) and (3.8). Let us assume that for any $j \in N_{r+1}$ a first time $t = t(r_j) < +\infty$

exists at which $y_j(t,\ r_j) \in X(t;\ G_2,\ V)$.

Let $t^* = \min_{j \in N_{r+1}} t(r_j)$. Appealing to the construction of the evaders' controls, Lemmas 1 and 2 and the induction hypothesis, we conclude that evasion is possible in game (2.1) from the initial state $z(t^*)$. This proves the theorem.

Corollary 5. Suppose that in game (2.1) V is a strictly convex compact set with smooth boundary, $n = 2k$, $m = k$, the initial state z^0 is such that $y_s^0 \neq y_q^0$ for any $s,\ q \in N_k,\ s \neq q$, and pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{2k}$ exist such that

$$y_j^0 \in \operatorname{int}\operatorname{co}\left\{x_{i_1}^0, \ldots, x_{i_{k+1}}^0\right\},\quad j = 1, \ldots, k$$

Then the evasion problem is solvable from the initial state z^0.

Proof. Let $y_j^0 \notin \operatorname{int}\operatorname{co}\{x_1^0, \ldots, x_{k+1}^0\}$, $j = 1, \ldots, k$. Consider the sets $G_1 = \operatorname{co}\{x_{k+2}^0, \ldots, x_{2k}^0\}$, $G_2 = \operatorname{co}\{x_1^0, \ldots, x_{k+1}^0\}$. If $j \in N_k$ exists for which $y_j^0 \in G_1$, then by Corollary 4 evasion is possible from the initial state z^0. But if $y_j^0 \notin G_1 (j = 1, \ldots, k)$, then all the assumptions of Theorem 2 are satisfied for the initial state z^0.

Remark 1. Let $x_i (i = 1, \ldots, n,\ n \geq k+2)$, $y_j (j = 1, \ldots, m,\ m \leq k)$ be given points in R^k, and suppose that for any pairwise distinct indices $i_i, \ldots, i_{k+1} \in N_n$ an index $j \in N_m$ exists such that

$$y_j \in \operatorname{int}\operatorname{co}\left\{x_{i_1}, \ldots, x_{i_{k+1}}\right\}$$

Then

$$x_{i_{k+2}} \notin \operatorname{co}\left\{x_{i_1}, \ldots, x_{i_{k+1}}\right\}$$

for any pairwise distinct $i_1, \ldots, i_{k+2} \in N_n$.

Remark 2. If $x_i (i = 1, \ldots, k+2)$, y_1, y_2 are given points in R^k $(k \geq 3)$, then pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{k+2}$ exist such that

$$y_j \notin \operatorname{int}\operatorname{co}\left\{x_{i_1}, \ldots, x_{i_{k+1}}\right\},\quad j = 1,2 \tag{3.13}$$

Proof. Suppose the contrary: for any pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{k+2}$, $j \in N_2$ exists such that

$$y_j \in \operatorname{int}\operatorname{co}\left\{x_{i_1}, \ldots, x_{i_{k+1}}\right\}$$

Without loss of generality, we may assume that

$$y_1 \in \operatorname{int}\operatorname{co}\left\{x_1, \ldots, x_{k+1}\right\}$$
$$y_2 \in \operatorname{int}\operatorname{co}\left\{x_1, \ldots, x_{k+2}\right\} \setminus \operatorname{co}\left\{x_1, \ldots, x_{k+1}\right\}$$

We represent the simplex $B = \operatorname{co}\{x_1, \ldots, x_{k+1}\}$ as the intersection of $k+1$ closed half-spaces $\overline{H}_i^+$. Since $x_{k+2} \notin B$, there is at least one half-space $\overline{H}_i^+$ that does not contain x_{k+2}. Let

$$x_{k+2} \in \bigcup_{r=l}^{k+1} \overline{H}_r^+ \setminus \left(\bigcup_{r=1}^{l} \overline{H}_r^+\right),\ l \geq 1$$

If $l > 1$, we consider l k-dimensional simplexes, each of which is the convex hull of x_{k+2} and the $(k-1)$-dimensional principal face of the simplex B in the hyperplane H_r, $r \in N_l$. Here H_r is the hyperplane bounding $\overline{H}_r^+$. Clearly, the interiors of these l simplexes do not intersect one another and the interior of the set $\operatorname{co}\{x_1, \ldots, x_{k+2}\}$ must contain the initial positions of at least $l+1$ evaders.

Let $l = 1$. We may assume that H_1 passes through the points $x_1, \ldots, x_k$ and $x_{k+1} \in \overline{H}_1^+$. Consider the

k-dimensional simplexes

$$A_1 = \mathrm{co}\{x_2, x_3, \ldots, x_k, x_{k+1}, x_{k+2}\}$$
$$A_2 = \mathrm{co}\{x_1, x_3, \ldots, x_k, x_{k+1}, x_{k+2}\}$$
$$\vdots$$
$$A_k = \mathrm{co}\{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, x_{k+2}\}$$

We shall show that the interiors of any two of these simplexes are disjoint. For example, let us consider A_1 and A_2 and find a hyperplane separating them. To do this, we consider the cone defined as the union of all rays emanating from the point x_{k+1} and passing through points of the set $\mathrm{co}\{x_1^0, \ldots, x_k^0\}$. The interior of this cone contains the point x_{k+2}^0. Hence the hyperplane H through the points $x_3^0, x_4^0, \ldots, x_{k+1}^0$, x_{k+2}^0 intersects the interior of the cone and, therefore, the points x_1^0, x_2^0 lie on different sides of H. Clearly, H separates A_1 and A_2. Thus the interior of $\mathrm{co}\{x_1^0, \ldots, x_{k+2}^0\}$ contains the initial positions of at least k evaders. This is a contradiction.

Thus, whatever the points x_i $(i=1, \ldots, k+2)$, y_1^0, y_2^0 in R^k $(k \geq 3)$, there are always pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{k+2}$ such that condition (3.13) holds.

4. Now, using the conditions established above for the local evasion problem to be solvable, let us consider the global evasion problem in some specific cases.

Theorem 3. Suppose that in game (2.1) V is a strictly convex compact set with smooth boundary, $n=k+2$ and $m=2$.

Then the evasion problem is solvable.

Proof. Let $z^0 = (x_1^0, \ldots, x_{k+2}^0, y_1^0, y_2^0)$ be some arbitrarily chosen initial state. We may assume without loss of generality that $y_1^0 \neq y_2^0$. Let us assume that $k=2$ and

$$y_j^0 \in \mathrm{int}\,\mathrm{co}\{x_1^0, \ldots, x_4^0\}, \quad j=1,2$$

If pairwise distinct indices exist i_1, i_2, $i_3 \in N_4$ such that

$$y_j^0 \notin \mathrm{int}\,\mathrm{co}\{x_{i_1}^0, x_{i_2}^0, x_{i_3}^0\}, \quad j=1,2$$

then by Corollary 5 the evasion problem is solvable. Hence, in particular, it follows that if the initial positions of any three players are collinear, evasion is possible from the initial state. This assertion also follows from Corollary 4.

Suppose that for any pairwise distinct indices i_1, i_2, $i_3 \in N_4$ $j \in N_2$ exists such that $y_j^0 \in \mathrm{int}\,\mathrm{co}\{x_{i_1}^0, x_{i_2}^0, x_{i_3}^0\}$. If

$$y_1^0 \in \mathrm{int}\,\mathrm{co}\{y_2^0, x_{i_1}^0, x_{i_2}^0\}, \quad i_1, i_2 \in N_4 \tag{4.1}$$

then it is readily seen that

$$y_2^0 \in \mathrm{int}\,\mathrm{co}\{y_1^0, x_{i_3}^0, x_{i_4}^0\}, \quad i_3, i_4 \in N_4 \setminus \{i_1, i_2\} \tag{4.2}$$

Thus N_4 can be partitioned into two disjoint sets $I_1 = \{i_1, i_2\}$, $I_2 = \{i_3, i_4\}$ for which (4.1) and (4.2) hold.

At the initial time, choose two sets F_1, $F_2 \in \mathrm{co}\Omega(R^k)$ such that

$$x_{i_l}^0, x_{i_{l+1}}^0 \in \mathrm{int}\,F_j, \quad l=2j-1, \quad y_j^0 \in \partial F_j, \quad j=1,2 \tag{4.3}$$

and vectors $r_j \in P(F_j, y_j^0)(j=1, 2)$ exist for which

$$y_1(t,r_1) \neq y_2(t,r_2) \text{ for } t \geq 0 \tag{4.4}$$

where $y_j(t, r_j)$ is the trajectory of player E_j corresponding to the control $v_j(t)$ chosen on the basis of (3.6), in which $\psi_j(t, r_j)$ is a solution of system (1.2) corresponding to the initial condition $\psi_j(0)=r_j$. Fix vectors $r_j \in P(F_j, y_j^0)$ satisfying inequality (4.4).

Let $t(r_j)$ denote the first time at which $y_j(t, r_j) \in X(t; F_l, V)$, $l \in N_2 \setminus \{j\} (j=1, 2)$. If $q \in N_2$ exists such that $y_q(t, r_q) \notin X(t, F_l, V)$, $l \in N_2 \setminus \{q\}$ for all $t \geq 0$, then player E_q can evade capture. We will therefore assume that $t(r_j) < +\infty (j=1, 2)$ we may assume without loss of generality that $t(r_1) \leq t(r_2)$.

Up to the first time $t' \in (0, t(r_1))$ at which the three players are situated on a single straight line, the control $v_j(t) (j=1, 2)$ is determined from (3.6). Such a time t' exists, because

$$y_j(t(r_1),r_j) \in \partial X(t(r_1); F_2,V), \ j=1,2$$

$$x_{i_3}(t(r_1)), \ x_{i_4}(t(r_1)) \in \operatorname{int} X(t(r_1); F_2,V)$$

for any controls $u_{i_3}(t)$, $u_{i_4}(t)$ in the interval $[0, t(r_1)]$, and therefore

$$y_2(t(r_1), r_2) \notin \operatorname{co}\{y_1(t(r_1),r_1), \ x_{i_3}(t(r_1)), \ x_{i_4}(t(r_1))\}$$

At a time t', by Corollary 4, the evasion problem from the state $z(t')=(x_1(t'), \ldots, x_4(t'), y_1(t'), y_2(t'))$ is solvable.

If $k > 2$, the solvability of the global evasion problem in game (2.1) with $n=k+2$, $m=2$ follows from Remark 2. This completes the proof of the theorem.

Theorem 4. Suppose that in game (2.1) V is a strictly convex compact set with smooth boundary, $n=2k$ and $m=k$.

Then the global evasion problem is solvable.

Proof. The assertion has already been proved for $k=2$. Suppose now that $k \geq 3$ and let $z^0=(x_1^0, \ldots, x_{2k}^0, y_1^0, \ldots, y_k^0)$ be an initial state. If pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{2k}$ exist such that

$$y_j^0 \notin \operatorname{int} \operatorname{co}\{x_{i_1}^0,\ldots,x_{i_{k+1}}^0\}, \ j=1,\ldots,k \tag{4.5}$$

then by Corollary 5 the evasion problem is solvable. Let us assume that for any pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{2k}$ there is a $j \in N_k$ such that

$$y_j^0 \in \operatorname{int} \operatorname{co}\{x_{i_1}^0,\ldots,x_{i_{k+1}}^0\}$$

It follows from Remark 1 that

$$x_{i_{k+2}}^0 \notin \operatorname{co}\{x_{i_1}^0,\ldots,x_{i_{k+1}}^0\}$$

for any pairwise distinct indices $i_1, \ldots, i_{k+2} \in N_{2k}$. Then for any $l \in N_k$

$$x_{k+l}^0 \notin \operatorname{co}\{x_1^0,\ldots,x_{k+l-1}^0\}$$

an index $j_l \in N_k$ exists such that

$$y_{j_l}^0 \in \operatorname{int} \operatorname{co}\{x_1^0,\ldots,x_{k+l}^0\} \setminus \operatorname{co}\{x_1^0,\ldots,x_{k+l-1}^0\}$$

Since there are k evaders participating in the game, the set int co $\{x_1^0, \ldots, x_{k+l}^0\}$ will contain the initial positions of exactly l evaders. Moreover, for any pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{k+l}$ $j \in \{j_1, \ldots, j_l\}$ exists such that

$$y_j^0 \in \operatorname{int} \operatorname{co}\left\{x_{i_1}^0, \ldots, x_{i_{k+1}}^0\right\}$$

But already when $l=2$ it follows from Remark 2 that pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{k+2}$ exist such that $y_{j_l}^0 \notin \operatorname{int} \operatorname{co}\{x_{i_1}^0, \ldots, x_{i_{k+1}}^0\}$, $l=1, 2$. We have arrived at a contradiction.

Consequently, for any initial state z^0 pairwise distinct indices $i_1, \ldots, i_{k+1} \in N_{2k}$ exist for which condition (4.5) holds. The theorem is proved.

An upper bound has been established for the minimum number of evaders in a game with n pursuers such that the global evasion problem is solvable in the case of simple motion [2]. In the case of game (2.1), imposing fairly weak conditions on the domains of the values of the players' controls, one can show that the same bound holds. Let $[a]$ denote the integer part of a number a.

Theorem 5. Suppose that in game (2.1) V is a strictly convex compact set with smooth boundary. If $n \geqslant 2$, $m \geqslant (p+1)2^{p+1}+2$, where $p=[\log_2(n-1)]$, then the global evasion problem is solvable.

REFERENCES

1. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
2. PETROV N. N. and PETROV N. N., On a differential game of "cops and robbers". *Differents. Uravn.* **19**, 8, 1366–1374, 1983,
3. GRIGORENKO N. L., The evasion problem in multi-person differential games. *Mat. Sbornik*. **135**, 1, 36–45, 1988.
4. CHIKRII A. A. and PROKOPOVICH P. V., On the evasion problem in the interaction of groups of moving objects. *Kibernetika* 5, 59–63, 78, 1989.
5. CHIKRII A. A. and PROKOPOVICH P. V., The problem of the interaction of groups of controllable objects. In: *Problems of Progressive Control and Construction of Extensions*, pp. 60–69. Ural. Otd. Akad. Nauk SSSR, Sverdlovsk, 1990.
6. CHIKRII A. A., *Controllable Conflict Processes*. Naukova Dumka, Kiev, 1992.
7. BLAGODATSKIKH V. I., *Theory of Differential Inclusions*, Pt. 1. Izd. Mosk. Gos. Univ., Moscow, 1978.
8. BLAGODATSKIKH V. I., *Linear Theory of Optimal Control*. Izd. Mosk. Gos. Univ., Moscow, 1978.
9. PSHENICHNYI B. N., Simple pursuit by several objects. *Kibernetika* 3, 145–146, 1976.
10. CHIKRII A. A. and PROKOPOVICH P. V., On the problem of simple pursuit of one evader by a group. *Kibernetika i Sistemnyi Analiz* 3, 131–137, 1992.

Translated by D.L.